QUASICONFORMAL EXTENSIONS OF HARMONIC MAPPINGS

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Abstract. We derive a very general condition for a sense-preserving harmonic mapping with dilatation a square to be injective in the unit disk $\mathbb{D}$ and to admit a quasiconformal extension to the extended complex plane. The analysis depends on geometric properties of an extension of the Weierstrass-Enneper lift to the extended plane that glues the parametrized minimal surface to a complementary topological hemisphere. The resulting topological sphere renders an entire graph over the complex plane provided additional restriction on the dilatation are satisfied. The projection results in the desired extension. Several corollaries are drawn from the general criterion.

1. Introduction

The purpose of this paper is to derive injectivity criteria and sufficient conditions for quasiconformal extensions of harmonic mappings defined in the unit disk $\mathbb{D}$. We borrow this classical theme from geometric function theory, where considerable work can be found in the form of criteria that depend on the Schwarzian derivative. The conditions we obtain here are expressed in terms of a Schwarzian derivative for harmonic mappings, and can be considered a refinement of a recent general result for the injectivity of the Weierstrass-Enneper lift found in [6], as well as a generalization of the main result in [14]. We will appeal to a geometric construction in [6] that sets up circle fibrations of $\mathbb{R}^3$ in domain and range that are paired in a natural way by means of the lift (see also [14]). Aside from certain extremal configurations, the Weierstrass-Enneper lifts satisfying the criterion in [6] are injective in $\mathbb{D}$, and admit a continuous extension the closed disk that remains injective. In other words, the boundary of the parametrized minimal surface is a simple closed curve. The fibration of space in the range allows for an extension of the lift by reflecting the minimal surface across its boundary. The resulting topological sphere renders an entire graph over the complex plane provided additional restrictions on the dilatation of the harmonic mapping are satisfied. This is the heart of the present paper and requires estimating the differential of the reflection.

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The projection of the graph onto $\mathbb{C}$ will result in the desired extension of the harmonic mapping. As a byproduct, the harmonic mapping will be univalent in $\mathbb{D}$, and the explicit extensions obtained appear as natural generalizations of classical conditions for holomorphic mappings ([19, 2, 4, 18, 3, 5], to mention some).

The paper is organized as follows. In the remainder of the Introduction we give a brief account of the main facts about harmonic mappings and Weierstrass-Enneper lifts. At the end of this section we will state the main result of this paper. In Section 2, we will summarize the results from [6] that will be required and will set up the extension of the lift as a homeomorphism of the extended complex plane onto a topological sphere in $\mathbb{R}^3 \cup \{\infty\}$. In Section 3 we study the issue of quasiconformality of the extension of the lift as a mapping from $\mathbb{C}$ into $\mathbb{R}^3$, leaving the proof of the main theorem for Section 4. The final section will be devoted to drawing various corollaries.

A planar harmonic mapping is a complex-valued harmonic function $f(z)$, $z = x+iy$, defined on some domain $\Omega \subset \mathbb{C}$. If Ω is simply connected, the mapping has a canonical decomposition $f = h+\overline{g}$, where h and g are analytic in Ω and $g(z_0) = 0$ for some specified point $z_0 \in \Omega$. The mapping f is locally univalent if and only if its Jacobian $|h'|^2 - |g'|^2$ does not vanish. It is said to be orientation-preserving if $|h'(z)| > |g'(z)|$ in Ω, or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property $|\omega(z)| < 1$ in Ω.

According to the Weierstrass–Enneper formulas, a harmonic mapping $f = h+\overline{g}$ with $|h'(z)| + |g'(z)| \neq 0$ lifts locally to map into a minimal surface, Σ, described by conformal parameters if and only if its dilatation $\omega = q^2$, the square of a meromorphic function q. The Cartesian coordinates (U, V, W) of the surface are then given by

$$U(z) = \operatorname{Re}\{f(z)\}, \quad V(z) = \operatorname{Im}\{f(z)\}, \quad W(z) = 2\operatorname{Im}\left\{\int_{z_0}^{z} h'(\zeta)q(\zeta)\, d\zeta\right\}.$$

We use the notation

$$\widetilde{f}(z) = \big(U(z), V(z), W(z)\big)$$

for the lifted mapping of Ω into Σ. The height of the surface can be expressed more symmetrically as

$$W(z) = 2\operatorname{Im}\left\{\int_{z_0}^{z} \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta\right\},$$

since a requirement equivalent to $\omega = q^2$ is that $h'g'$ be the square of an analytic function. The first fundamental form of the surface is $ds^2 = e^{2\sigma}|dz|^2$, where the conformal factor is

$$e^{\sigma} = |h'| + |g'|.$$

The Gauss curvature of the surface at a point $\widetilde{f}(z)$ for which $h'(z) \neq 0$ is

$$K = -e^{-2\sigma}\Delta\sigma = -\frac{4|q'|^2}{|h'|^2(1+|q|^2)^4}\,, \tag{1.1}$$

where Δ is the Laplacian operator. Further information about harmonic mappings and their relation to minimal surfaces can be found in [17].

For a harmonic mapping $f = h + \overline{g}$ with $|h'(z)| + |g'(z)| \neq 0$, whose dilatation is the square of a meromorphic function, we have defined [10] the *Schwarzian derivative* by the formula

$$\mathcal{S}f = 2\big(\sigma_{zz} - \sigma_z^2\big)\,, \tag{1.2}$$

where

$$\sigma_z = \frac{1}{2}\left(\sigma_x - i\sigma_y\right)\,.$$

Some background for this definition is discussed in Section 2. With $h'(z) \neq 0$ and $g'/h' = q^2$, a calculation (*cf.* [10]) produces the expression

$$\mathcal{S}f = \mathcal{S}h + \frac{2\overline{q}}{1+|q|^2}\left(q'' - \frac{q'h''}{h'}\right) - 4\left(\frac{q'\overline{q}}{1+|q|^2}\right)^2\,.$$

The formula remains valid if ω is not a perfect square, provided that neither h' nor g' has a simple zero.

In our main result stated below, we consider a complete metric of negative curvature $e^{\rho}|dz|$ in $\mathbb{D}$, and two generic conditions for the boundary $\partial\mathbb{D}$ at infinity to be *visible*. These conditions (ULP) and (BPJ) are defined in Section 2, and state, briefly, that geodesics have a unique limit point in $\partial\mathbb{D}$, and can be chosen to join arbitrary pairs of points on the boundary.

Theorem 1.1. *Let $f = h + \bar{g}$ be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$, and let $e^{\rho}|dz|$ be a complete metric of negative curvature in $\mathbb{D}$ satisfying (ULP) and (BPJ). Suppose that*

$$\left|\mathcal{S}f - 2\left(\rho_{zz} - \rho_z^2\right)\right| + e^{2\sigma}|K| \leq 2t\rho_{z\bar{z}} \tag{1.3}$$

for some $0 \leq t < 1$, that

$$|\rho_z - \sigma_z| \leq C\sqrt{\rho_{z\bar{z}}} \tag{1.4}$$

for some constant C, and that

$$\sup_{z\in\mathbb{D}} \sqrt{|\omega(z)|} < \frac{2s}{1+\sqrt{1+4s^2}}\ \ ,\ \ s = \frac{1-t}{2\sqrt{2}C\sqrt{t}}\,. \tag{1.5}$$

Then f is injective and has a quasiconformal extension to $\mathbb{C}$ given by

$$\mathcal{E}_f(z) = \begin{cases} f(z)\,, \ z \in \overline{\mathbb{D}} \\ \\ f(z^*) + \dfrac{h'(z^*)}{(\rho-\sigma)_z(z^*)} + \dfrac{\overline{g'}(z^*)}{(\rho-\sigma)_{\bar z}(z^*)}\,,\ z^* = \dfrac{1}{\bar z}\,,\ z \notin \mathbb{D}. \end{cases} \tag{1.6}$$

2. Weierstrass-Enneper Lifts, Circle Bundles and the Reflection

In this section, we collect various results from [6] that will be required in the present paper. We shall consider metrics $\mathbf{g_1} = e^{2\rho}\mathbf{g_0}$ in $\mathbb{D}$ conformal to the Euclidean metric $\mathbf{g_0}$, required to be complete and of negative curvature. The general criterion established there takes the following form for metrics of this type.

Theorem A. *Let f be a harmonic mapping with dilatation $\omega = q^2$ the square of a meromorphic function in $\mathbb{D}$. Let $\mathbf{g_1} = e^{2\rho}\mathbf{g_0}$ be a complete metric of non-positive curvature. If*

$$\left|\mathcal{S}f - 2\left(\rho_{zz} - \rho_z^2\right)\right| + e^{2\sigma}|K| \leq 2\rho_{z\bar z} \tag{2.1}$$

then the lift $\widetilde{f}$ is injective in $\mathbb{D}$.

The proof relies on an application of Ahlfors' Schwarzian for curves and its interplay with the (conformal) Schwarzian for the harmonic mapping. To be more precise, injectivity follows from a general condition for curves to be simple established in [7], and an adequate bound for the Schwarzian of the curves obtained from the restriction of $\widetilde{f}$ to arbitrary geodesics in $\mathbb{D}$. The bound is guaranteed by (2.1), and will hold for $\widetilde{f}$ as well as for any Möbius shift $T \circ \widetilde{f}$ because of the invariance of Ahlfors' Schwarzian under the conformal group.

In order to study the behavior of the lift $\widetilde{f}$ near the boundary $\partial\mathbb{D}$ it became necessary to impose some generic conditions on the metric to ensure sufficient control of the ends of geodesics. The conditions (ULP) and (BPJ) defined below will be assumed to hold for the metric $\mathbf{g_1}$.

Definition 2.1. The metric $\mathbf{g_1}$ on $\mathbb{D}$ has the *Unique Limit Point* property (ULP) if:

(a) Let $z_0 \in \mathbb{D}$. If $\gamma(t)$, $0 \leq t < \infty$ is a maximally extended geodesic starting at z_0 then $\lim_{t\to\infty}\gamma(t)$ exists (in the Euclidean sense). We denote it by $\gamma(\infty) \in \partial\mathbb{D}$.

(b) The limit point is a continuous function of the initial direction at z_0.

(c) For any $\zeta \in \partial\mathbb{D}$ there is a geodesic starting at z_0 whose limit point is ζ.

We say a little more about part (c) in this condition. The assumption of negative curvature implies that the limit point is a monotonic function of the initial direction at the base point. Part (b) requires that it is continuous. It is conceivable that, for some metrics, all geodesics from a base point might tend to the same limit point on the boundary, so the mapping from initial directions to points on $\partial\mathbb{D}$ would reduce to a constant. We want to avoid this degenerate situation and be certain that every boundary point is *visible*, so we include that fact in the statement of (ULP). (ULP) is also important because it ensures that lifts satisfying Theorem A admit a (spherically) continuous extensions to the closed disk.

Definition 2.2. The metric $\mathbf{g_1}$ on $\mathbb{D}$ has the *Boundary Points Joined* property (BPJ) if any two points on $\partial\mathbb{D}$ can be joined by a geodesic which lies in $\mathbb{D}$ except for its endpoints.

A lift satisfying Theorem A was called *extremal* if the extension to the closed disk was not injective. It was shown in [6] that for extremal lifts equality had to hold in (2.1) along a geodesic joining the points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ for which $\widetilde{f}(\zeta_1) = \widetilde{f}(\zeta_2)$. In particular, if for example, condition (2.1) is satisfied with strict inequality everywhere then $\widetilde{f}$ cannot be extremal.

A fundamental tool underlying the analysis is the function

$$u_{\widetilde{f}}(z) = \sqrt{e^{\rho-\sigma}}\,,$$

which becomes convex relative to $\mathbf{g_1}$ when (2.1) is in force [6, Lemma 4.1]. Convexity is obtained from the estimates derived for Ahlfors' Schwarzian, and it follows from the invariance of this operator under the Möbius group that, even though $(T\circ\widetilde{f})(\mathbb{D})$ will in general not minimal, the function $u_{T\circ\widetilde{f}}$ associated with the conformal immersion $T\circ\widetilde{f}$ will remain convex.

We set up circle fibrations of space in domain and range, with a pairing induced by the lift $\widetilde{f}$. As a general configuration, let B be a smooth, open surface in $\mathbb{R}$, and consider a family $\mathfrak{C}(B)$ of Euclidean circles C_p indexed by $p \in B$, at most one of which is a Euclidean line, having the properties:

(i) C_p is orthogonal to B at p and $C_p \cap \overline{B} = \{p\}$;

(ii) if $p_1 \neq p_2$ then $C_{p_1} \cap C_{p_2} = \emptyset$;

(iii) $\bigcup_{p\in B} C_p = \mathbb{R}^3 \setminus \partial B$.

We regard the point at ∞ as lying on the line in $\mathfrak{C}(B)$. We refer to $p \in C_p$ as the base point. If B is unbounded then there is no line in $\mathfrak{C}(B)$, for a line would meet $\overline{B}$ at its base point and at the point at infinity, contrary to (i).

The model case is $B = \mathbb{D}$, with $\mathfrak{C}_\mathfrak{o} = \mathfrak{C}(\mathbb{D})$ being the collection of circles orthogonal to the complex plane passing through $z \in \mathbb{D}$ and its reflection $1/\bar{z}$. In this case, only the circle through the origin becomes a line. Based on the following lemma, it was shown in [6] that such a bundle of circles could be set up in the image with base $B = \widetilde{f}(\mathbb{D})$. The circles appear as the set of points $q \in \mathbb{R}^3$ where the Möbius inversion

$$T(w) = \frac{w-q}{||w-q||^2}$$

produces a critical point of the function $u_{T\circ\widetilde{f}}$ at a specified point in $\mathbb{D}$. One is justified to limit the attention to inversions alone (instead of the entire class of Möbius transformations) because the critical point is unchanged when T is just affine. We will denote the above inversion by I_q.

Lemma A. *Let $\widetilde{f}$ satisfy (2.1) and let $z_0 \in \mathbb{D}$ be fixed. Consider the set $C = C_{z_0}$ of points $q \in \mathbb{R}^3$ for which $u_{I_q\circ\widetilde{f}}$ has a critical point at z_0. Then*

(i) C is a circle orthogonal to Σ at $\widetilde{f}(z_0)$ with radius $r(z_0) = \dfrac{e^{\sigma(z_0)}}{2|\nabla \log u_{\widetilde{f}}(z_0)|}$;

(ii) C is symmetric with respect to the tangent plane to Σ at $\widetilde{f}(z_0)$;

(iii) $(C \setminus \{\widetilde{f}(z_0)\}) \cap \overline{\Sigma} = \emptyset$.

The family of circles $C_z, z \in \mathbb{D}$, was shown to be a fibration of space with base Σ, meaning in particular, that the circles are disjoint for different base points. The extension of $\widetilde{f}$ as a homeomorphism of $\mathbb{C}$ onto a topological sphere in 3-space will will glue Σ to the surface Σ^* obtained by intersecting the fibers $C_z/\{\widetilde{f}(z)\}$ with the tangent plane to Σ at $w = \widetilde{f}(z)$. The construction is obviously continuous on Σ, and from (ii), (iii) above, the point of intersection $w^* = \mathcal{R}(w)$ lies outside $\overline{\Sigma}$ and is diametrically opposite to w on C_z. The injectivity of the reflection is guaranteed because the circles are pairwise disjoint. From (i) and based on the convexity of $u_{\widetilde{f}}$, it was shown that $\mathcal{R}$ is (spherically) continuous on $\partial\Sigma$. The extension of $\widetilde{f}$ defined by

$$\tilde{\mathcal{E}}_{\widetilde{f}}(z) = \begin{cases} \widetilde{f}(z) & , \quad |z| \le 1 \\ \mathcal{R}(\widetilde{f}(\frac{1}{\bar{z}})) & , \quad |z| > 1 \end{cases} \tag{2.2}$$

provides therefore a homeomorphism of $\mathbb{C} \cup \{\infty\}$ onto a topological sphere in $\mathbb{R}^3 \cup \{\infty\}$.

In order to study the quasiconformal distortion of $\widetilde{F}$ and later on, of its projection onto $\mathbb{C}$, we must derive explicit formulas for $\mathcal{R} = \mathcal{R}(w)$. We will first determine an equation defining a given circle C_{z_0}. Because

$$||D(I_q \circ \widetilde{f})|| = ||\widetilde{f} - q||^{-2} e^{\sigma} ,$$

it follows that

$$2 \log u_{I_q\circ\widetilde{f}} = \rho - \sigma + 2 \log ||\widetilde{f} - q|| ,$$

hence the critical point condition on $u_{I_q\circ\widetilde{f}}$ is given by the equation

$$\rho_z = \sigma_z + \frac{1}{||\widetilde{f}-q||^2}\left(\langle \widetilde{f}_x, q-\widetilde{f}\rangle - i\langle \widetilde{f}_y, q-\widetilde{f}\rangle\right), \tag{2.3}$$

where $\langle\, ,\, \rangle$ denotes the Euclidean inner product. We seek the point $w^* = q$ satisfying (2.2) that lies in the tangent plane to Σ at $\widetilde{f}$. If we write

$$w^* = \widetilde{f} + a\widetilde{f}_x + b\widetilde{f}_y$$

we find that

$$a + ib = \frac{1}{\rho_z - \sigma_z}.$$

Therefore

$$w^* = w + \frac{\alpha e^{\sigma}}{\alpha^2+\beta^2}X + \frac{\beta e^{\sigma}}{\alpha^2+\beta^2}Y, \tag{2.4}$$

where $X = e^{-\sigma}\widetilde{f}_x$, $Y = e^{-\sigma}\widetilde{f}_y$ are unit tangent vectors to Σ, and $\alpha + i\beta = (\rho-\sigma)_{\bar z}$. We finally rewrite (2.3) purely in terms of quantities on Σ. We consider on Σ the conformal metric $\lambda_\Sigma|dw|$ defined by

$$\lambda_\Sigma(\widetilde{f})e^{\sigma} = e^{\rho}, \tag{2.5}$$

so that $\widetilde{f} : (\mathbb{D}, e^{\rho}|dz|) \to (\Sigma, \lambda_\Sigma|dw|)$ becomes an isometry. Then (2.3) translates to

$$\mathcal{R}(w) = w + 2J(\nabla \log \lambda_\Sigma), \tag{2.6}$$

where $J = I_0$ is the inversion centered at the origin and ∇ is the Euclidean gradient operator on Σ.

3. Quasiconformal Distortion

The purpose in this section is to study the quasiconformal distortion of the reflection $\mathcal{R}$. We will show that

$$m(w) \le ||\overline{D}_V\mathcal{R}|| \le M(w),$$

where $\sup_{w\in\Sigma} M(w)/m(w)$ is bounded by a quantity depending on t and C, and $\overline{D}_V\mathcal{R}$ stands for the derivative of $\mathcal{R}$ direction V tangent to Σ at a given point. Necessarily the analysis shifts to Σ and some of the geometric notions attached to Σ as a surface in $\mathbb{R}^3$ with its induced Euclidean metric $\mathbf{g}_0$, e.g., the gradient and the Hessian of a function, the covariant derivative and second fundamental form, and the curvature. If V is a vector field on Σ we let $\overline{D}_V$ be the Euclidean covariant derivative on $\mathbb{R}^3$ in the direction V, applied to a function or a vector field on Σ, and we let D_V be the covariant derivative on Σ. If ψ is a function on Σ then $\overline{D}_V\psi = D_V\psi = V\psi$. The gradient of ψ is the vector field defined by

$$\langle \nabla\psi, V\rangle = V\psi$$

and its Hessian is the symmetric, covariant 2-tensor defined by

$$\operatorname{Hess}\psi(V,W) = \langle D_V \nabla\psi, W\rangle.$$

If W is a vector field on Σ then

$$\overline{D}_V W = D_V W + I\!I(V,W)$$

where $I\!I(V,W)$ is the second fundamental form of Σ.

Next, we must formulate the inequality (1.3) to one for functions defined on the surface. This requires the full differential-geometric definition of the conformal Schwarzian as a symmetric, traceless 2-tensor, and uses in particular a generalization of the chain rule for the Schwarzian. For a function ψ defined on a 2-dimensional Riemannian manifold $(M, \mathbf{g})$ the Schwarzian tensor of ψ is

$$B_{\mathbf{g}}(\psi) = \operatorname{Hess}_{\mathbf{g}}\psi - d\psi \otimes d\psi - \frac{1}{2}(\Delta_{\mathbf{g}}\psi - ||\nabla_{\mathbf{g}}\psi||^2_{\mathbf{g}})\mathbf{g} \tag{3.1}$$

where the Hessian, Laplacian, gradient, and norm are taken with respect to a Riemannian metric $\mathbf{g}$. The final term is the trace of $\operatorname{Hess}_{\mathbf{g}}\psi - d\psi\otimes d\psi$, so the full tensor is traceless. If f is a conformal mapping with conformal factor $e^{2\psi}\mathbf{g}$ then, by definition,

$$\mathcal{S}_{\mathbf{g}} f = B_{\mathbf{g}}(\psi)\,.$$

In the case of a harmonic map f and its lift

$$\widetilde{f} : (\mathbb{D}, \mathbf{g_1}) \to (\Sigma, \mathbf{g_0})\,,$$

with conformal factor $\widetilde{f}^*(\mathbf{g}_0) = e^{2\sigma}|dz|^2$ as before, we have

$$\mathcal{S}f = \mathcal{S}\widetilde{f} = B(\sigma),$$

with respect to the Euclidean metric. In canonical coordinates, $B(\sigma)$ is a matrix of the form

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix},$$

where

$$a + ib = 2(\sigma_{zz} - \sigma_z^2)$$

was taken as the definition of the harmonic Schwarzian. Here, and below, when a quantity is calculated with respect to the Euclidean metric we drop the subscript $\mathbf{g}_0$.

The quantities defining $B_{\mathbf{g}}(\psi)$ which depend on the metric obey a certain generalized chain rule when the metric changes *conformally*. It reads, in one form,

$$B_{\hat{\mathbf{g}}}(\psi - \rho) = B_{\mathbf{g}}(\psi) - B_{\mathbf{g}}(\rho)\,, \quad \hat{\mathbf{g}} = e^{2\rho}\mathbf{g}\,,$$

and in terms of conformal mappings, say $(M_1, \mathbf{g}) \xrightarrow{\phi} (M_2, \mathbf{h}) \xrightarrow{\psi} (M_3, \mathbf{k})$,

$$\mathcal{S}_{\mathbf{g}}(\psi \circ \phi) = \phi^*(\mathcal{S}_{\mathbf{h}}\psi) + \mathcal{S}_{\mathbf{g}}\phi\,.$$

From the last equation, if ψ and ϕ are inverse to each other then $\mathcal{S}_{\mathbf{g}}\phi = -\phi^*(\mathcal{S}_{\mathbf{h}}\psi)$.

Specializing to our case, we find the following. Recall from (2.5) the metric $\lambda_\Sigma^2\,\mathbf{g}_0$ with $\lambda_\Sigma \circ \widetilde{f} = e^{\rho-\sigma}$. Consider $\widetilde{f}\colon (\mathbb{D},\mathbf{g_1}) \to (\Sigma,\mathbf{g_0})$ as a conformal mapping with conformal factor $e^{2(\sigma-\rho)}\mathbf{g_1}$. We take the Schwarzian tensor of $\widetilde{f}$ with respect to $\mathbf{g_1}$:

$$\mathcal{S}_{\mathbf{g_1}}\widetilde{f} = B_{\mathbf{g_1}}(\sigma-\rho).$$

Similarly, if $\phi = \widetilde{f}^{-1}$ then $\phi\colon (\Sigma,\mathbf{g}_0) \to (\mathbb{D},\mathbf{g_1})$ is conformal with conformal factor λ_Σ^2. The Schwarzian tensor of ϕ is with respect to the Euclidean metric on Σ and

$$\mathcal{S}\phi = B(\log\lambda_\Sigma).$$

From the formulas above,

$$B(\log\lambda_\Sigma) = \mathcal{S}\phi = -\phi^*\mathcal{S}_{\mathbf{g_1}}\widetilde{f} = -\phi^*(B_{\mathbf{g_1}}(\sigma-\rho))\,,$$

while

$$B_{\mathbf{g_1}}(\sigma-\rho) = B(\sigma) - B(\rho)\,,$$

because of the chain rule above.

On the other hand, $\phi\colon (\Sigma,\mathbf{g_2}) \to (\mathbb{D},\mathbf{g})$ is an isometry for $\mathbf{g_2} = \lambda_\Sigma^2\,\mathbf{g}_0$, thus

$$||B(\log\lambda_\Sigma)||_{\mathbf{g_2}} = ||B_{\mathbf{g_1}}(\sigma-\rho)||_{\mathbf{g_1}} = e^{-2\rho}||B(\sigma)-B(\rho)||\,.$$

Since also

$$||B(\log\lambda_\Sigma)||_{\mathbf{g_2}} = \lambda_\Sigma^{-2}||B(\log\lambda_\Sigma)||$$

we find that

$$||B(\log\lambda_\Sigma)|| = e^{-2\sigma}||B(\sigma)-B(\rho)||\,.$$

In the final term $B(\sigma)$ is the harmonic Schwarzian while $B(\rho)$ is represented by $2(\rho_{zz}-\rho_z^2)$. Combining these with (1.3) we find

$$||B(\log\lambda_\Sigma)|| + |K| = e^{-2\sigma}|Sf - 2(\rho_{zz}-\rho_z^2)| + |K| \le 2te^{-2\sigma}\rho_{z\bar{z}} = 2t\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar{z}}\,.$$

We summarize these calculations in the following lemma.

Lemma 3.1. *If f satisfies (1.3) then*

$$||B(\log\lambda_\Sigma)|| + |K| \le 2t\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar{z}}\,. \tag{3.2}$$

We proceed with the computation of $\overline{D}_V\mathcal{R}$ using (2.6),

$$\mathcal{R} = \mathrm{Id} + 2J(\nabla\log\lambda_\Sigma),$$

and the formula for the differential of J

$$DJ(x) = \frac{1}{||x||^4}(||x||^2\mathrm{Id} - 2Q(x))\,,$$

where $Q(x)$ is the matrix (x_ix_j) [1]. We have, first,

$$\overline{D}_V\mathcal{R} = V + 2J'(\nabla\log\lambda_\Sigma)(\overline{D}_V\nabla\log\lambda_\Sigma),$$

and also the relation

$$\overline{D}_V\nabla\log\lambda_\Sigma = D_V\nabla\log\lambda_\Sigma + II(V,\nabla\log\lambda_\Sigma).$$

Hence

$$\overline{D}_V\mathcal{R} = V + \frac{2}{||\nabla \log \lambda_\Sigma||^4}\left\{||\nabla \log \lambda_\Sigma||^2\overline{D}_V\nabla \log \lambda_\Sigma - 2Q(\nabla \log \lambda_\Sigma)(\overline{D}_V\nabla \log \lambda_\Sigma)\right\} .$$

At this point it is convenient to simplify the notation somewhat. Let

$$\Lambda = ||\nabla \log \lambda_\Sigma||, \quad Q = Q(\nabla \log \lambda_\Sigma), \quad II = II(V, \nabla \log \lambda_\Sigma) .$$

Furthermore,

$$\mathrm{Hess}(\log \lambda_\Sigma)(V, W) = \langle D_V \nabla \log \lambda_\Sigma, W\rangle$$

so we identify the vector $D_V\nabla \log \lambda_\Sigma$ with the 1-tensor $\mathrm{Hess}(\log \lambda_\Sigma)(V, \cdot\,)$ and write

$$H = D_V \nabla \log \lambda_\Sigma .$$

The Schwarzian tensor will enter through the Hessian terms, but this is not immediate. The expression for $\overline{D}_V\mathcal{R}$ is then given by

$$\overline{D}_V\mathcal{R} = V + \frac{2}{\Lambda^2}\left\{H - \frac{2}{\Lambda^2}Q(H) + II - \frac{2}{\Lambda^2}Q(II)\right\} ,$$

where it will be important to identify the parts tangent and normal to Σ.

From the definition,

$$[Q(\nabla \log \lambda_\Sigma)]_{ij} = (\nabla \log \lambda_\Sigma)_i(\nabla \log \lambda_\Sigma)_j ,$$

hence for any vector X we have that

$$Q(X) = \langle \nabla \log \lambda_\Sigma, X\rangle \nabla \log \lambda_\Sigma$$

is always tangent to Σ. Finally, H is tangent to Σ while II is normal to Σ, and we may write

$$\overline{D}_V\mathcal{R} = V + \frac{2}{\Lambda^2}\left(W_1 + W_2\right) , \tag{3.3}$$

where

$$W_1 = H - \frac{2}{\Lambda^2}Q(H) - \frac{2}{\Lambda^2}Q(II) \tag{3.4}$$

is tangent to Σ and

$$W_2 = II \tag{3.5}$$

is normal to it. It is interesting to note that the tangent planes to Σ and $\mathcal{R}(\Sigma)$ at respective points will be the same if $II = 0$, that is, when $K = 0$.

To find the norm $||\overline{D}_V\mathcal{R}||^2$ we use that Q is symmetric and that $Q^2 = \Lambda^2 Q$. Hence

$$\langle Q(II), V\rangle = \langle II, Q(V)\rangle = \langle \nabla \log \lambda_\Sigma, V\rangle\langle II, \nabla \log \lambda_\Sigma\rangle = 0 .$$

With this, in expanding $||\overline{D}_V\mathcal{R}||^2$ a number of terms then drop out and, at length, we obtain

$$||\overline{D}_V\mathcal{R}||^2 = 1 + \frac{4}{\Lambda^2}\langle H, V\rangle + \frac{4}{\Lambda^4}\left\{||H||^2 - 2\langle Q(H), V\rangle + ||II||^2\right\} \tag{3.6}$$

where we have also used $||V|| = 1$.

Referring to the definition we have

$$B(\log \lambda_\Sigma) = \text{Hess}(\log \lambda_\Sigma) - d\log\lambda_\Sigma \otimes d\log\lambda_\Sigma - \frac{1}{2}(\Delta \log\lambda_\Sigma - ||\nabla \log\lambda_\Sigma||^2)\mathbf{g}_0 \,.$$

Evaluate $B(\log\lambda_\Sigma)(V, \cdot\,)$ and treat this 1-tensor as a vector, which, continuing the pattern of notation, we will denote by B. With these abbreviations note that (3.2) implies

$$||B|| + |K| \leq 2t\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar{z}} \,. \tag{3.7}$$

Next, in components the 2-tensor $d\log\lambda_\Sigma \otimes d\log\lambda_\Sigma$ is exactly $Q(\nabla\log\lambda_\Sigma)$, which we have denoted by Q. Finally we write

$$\mu = \frac{1}{2}(\Delta\log\lambda_\Sigma - ||\nabla\log\lambda_\Sigma||^2) = \frac{1}{2}(\Delta\log\lambda_\Sigma - \Lambda^2)\,.$$

for the trace. In these terms

$$H = B + Q(V) + \mu V.$$

and in (3.6),

$$\langle H, V\rangle = \langle B, V\rangle + \langle Q(V), V\rangle + \mu\,,$$

$$||H||^2 = ||B||^2 + \Lambda^2\langle Q(V), V\rangle + \mu^2 + 2\langle B, Q(V)\rangle + 2\mu\langle B, V\rangle + 2\mu\langle Q(V), V\rangle\,,$$

$$\langle Q(H), V\rangle = \langle H, Q(V)\rangle = \langle B, Q(V)\rangle + \Lambda^2\langle Q(V), V\rangle + \mu\langle Q(V), V\rangle\,.$$

Substitution results in a quite compact expression:

$$||\overline{D}_V\mathcal{R}||^2 = \frac{4}{\Lambda^4}\left\{||B + \frac{1}{2}(\Delta\log\lambda_\Sigma)V||^2 + ||II||^2\right\} = \frac{4}{\Lambda^4}\left\{||W_1||^2 + ||W_2||^2\right\}\,.$$

This is the penultimate form. The final step, to bring in the inequality (3.7) for the Schwarzian, is to introduce the curvature.

The curvature of Σ with the metric $\lambda_\Sigma^2\mathbf{g}_0$ is $-4e^{-2\rho}\rho_{z\bar{z}}$ since $(\Sigma, \lambda_\Sigma^2\,\mathbf{g}_0)$ is isometric to $(\mathbb{D}, \lambda_\mathbb{D}|dz|^2)$. For the curvature $K \leq 0$ of Σ as a minimal surface one obtains

$$\Delta\log\lambda_\Sigma = 4\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar{z}} - |K|.$$

Hence

$$||D_V\mathcal{R}||^2 = \frac{4}{\Lambda^4}\left\{||B - \frac{1}{2}|K|V + 2\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar{z}}V||^2 + ||II||^2\right\}. \tag{3.8}$$

We want to bound this from above and below.

To obtain a lower bound we drop the term $||II||^2$ and use (3.7):

$$||D_V\mathcal{R}|| \geq \frac{2}{\Lambda^2}||B - \frac{1}{2}|K|V + 2\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar{z}}V||$$

$$\geq \frac{2}{\Lambda^2}\left\{2\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar z} - || - B + \frac{1}{2}|K|V||\right\}$$

$$\geq \frac{2}{\Lambda^2}\left\{2\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar z} - ||B|| - \frac{1}{2}|K|\right\} \geq 4(1-t)\frac{\lambda_\Sigma^2}{\Lambda^2}e^{-2\rho}\rho_{z\bar z}\,. \tag{3.9}$$

To obtain an upper bound we have to estimate the term $||II||$. On a minimal surface we always have $II(X,Y) \leq \sqrt{|K|}||X||\,||Y||$, and so for our case

$$||II|| = ||II(V, \nabla \log \lambda_\Sigma)|| \leq \sqrt{|K|}\,||\nabla \log \lambda_\Sigma|| = \sqrt{|K|}\Lambda.$$

We need estimates for each of the factors on the right, and this is where we use (1.4), namely that

$$|\sigma_z - \rho_z| \leq C\sqrt{\rho_{z\bar z}}\,.$$

An inequality for the curvature follows simply from dropping the positive $||B||$ term in (3.7), giving

$$|K| \leq 2t\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar z}.$$

Next, from $\log(\lambda_\Sigma \circ \widetilde{f}) = \log \lambda_{\mathbb{D}} - \sigma$ and the bound on $|\sigma_z - \rho_z|$ we have

$$e^\sigma \Lambda = e^\sigma ||\nabla \log \lambda_\Sigma(\widetilde{f}(z))|| = 2|\sigma_z - \rho_z| \leq 2C\sqrt{\rho_{z\bar z}}\,.$$

Multiplying through by $e^{-\sigma}$ brings back λ_Σ on the right:

$$\Lambda \leq 2C\lambda_\Sigma\sqrt{e^{-2\rho}\rho_{z\bar z}}\,.$$

With this,

$$||II|| \leq \sqrt{|K|}\Lambda \leq 2\sqrt{2t}C\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar z}\,. \tag{3.10}$$

Back to the equation (3.8) for $||D_V\mathcal{R}||^2$, we have

$$\begin{aligned}||D_V\mathcal{R}|| &\leq \frac{2}{\Lambda^2}\left\{||B - \frac{1}{2}|K|V + 2\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar z}V|| + ||II||\right\}\\ &\leq \frac{2}{\Lambda^2}\left\{||B|| + \frac{1}{2}|K| + 2\lambda_\Sigma^2 e^{-2\rho}\rho_{z\bar z} + ||II||\right\}\\ &\leq 4\left(1 + t + \sqrt{2t}C\right)\frac{\lambda_\Sigma^2}{\Lambda^2}e^{-2\rho}\rho_{z\bar z}\,.\end{aligned}$$

Combining the upper and lower bounds for $||D_V\mathcal{R}||$ gives

$$\frac{\max_{||V||=1}||D_V\mathcal{R}||}{\min_{||V||=1}||D_V\mathcal{R}||} \leq \frac{1 + t + \sqrt{2t}C}{(1-t)}\,. \tag{3.11}$$

This shows that $\mathcal{R}$ is quasiconformal as a mapping from Σ to its reflection Σ^*. The extension of $\widetilde{f}$ to a mapping $\tilde{\mathcal{E}}_{\widetilde{f}} : \overline{\mathbb{C}} \longrightarrow \overline{\Sigma} \cup \Sigma^*$ is as in (2.2). It, too, is quasiconformal with the same bound for the distortion.

4. Quasiconformal Extension of Planar Harmonic mappings

In this section we consider the problem of injectivity and quasiconformal extension for the planar harmonic mapping $f = h + \bar{g}$ under the assumption that its lift $\widetilde{f}$ satisfies the hypotheses in Theorem 1.1. The method will be simply to project from $\overline{\Sigma} \cup \Sigma^*$ to the plane, and requires restricting the dilatation as in Theorem 1.1; the reward is the similarity of the resulting extension of the planar map with the classical conditions.

Lemma 4.1. *Suppose that $f = h+\bar{g}$ is locally injective with dilatation ω the square of an analytic function, and that $\widetilde{f}$ satisfies (1.3)-(1.6). If ω satisfies*

$$\sup_{z\in\mathbb{D}} \sqrt{|\omega(z)|} < \frac{2s}{1+\sqrt{1+4s^2}}\,, \quad s = \frac{1-t}{2\sqrt{2}C\sqrt{t}} \tag{4.1}$$

then Σ^* is locally a graph.

Proof. Fix a point $w = \widetilde{f}(z)$ on Σ. Let ϑ be the angle of inclination with respect to the vertical of the tangent plane $T_w(\Sigma)$. From the formulas for the components of $\widetilde{f}$, i.e., the formulas for the Weierstrass-Enneper lift, see [17], one can show that

$$\tan\vartheta = \frac{2\sqrt{|\omega(z)|}}{1-|\omega(z)|}. \tag{4.2}$$

For V a unit tangent vector to Σ at w we consider the tangential and normal components $D\tilde{\mathcal{E}}_{\widetilde{f}}(V)^{\top}$ and $D\tilde{\mathcal{E}}_{\widetilde{f}}(V)^{\perp}$ of $D\tilde{\mathcal{E}}_{\widetilde{f}}(V)$, expressed by the equation (3.3), (3.4) and (3.5). Then the angle of inclination of the tangent plane $T_{\mathcal{R}(w)}(\Sigma^*)$ to Σ^* at $\mathcal{R}(w)$ is

$$\vartheta + \tan^{-1}\frac{||D\tilde{\mathcal{E}}_{\widetilde{f}}(V)^{\perp}||}{||D\tilde{\mathcal{E}}_{\widetilde{f}}(V)^{\top}||}.$$

The surface Σ^* will be locally a graph if this angle is $< \pi/2$, and using (4.2) this condition can be written

$$\frac{2\sqrt{|\omega(z)|}}{1-|\omega(z)|}\,\frac{\|D\tilde{\mathcal{E}}_{\widetilde{f}}(V)^{\perp}\|}{\|D\tilde{\mathcal{E}}_{\widetilde{f}}(V)^{\top}\|} < 1. \tag{4.3}$$

Using (3.9) and (3.10) that estimate $||D_V\mathcal{R}||$ from below and $||II||$ from above, the requirement becomes

$$2\sqrt{2}C\frac{\sqrt{|\omega|}}{1-|\omega|}\,\frac{\sqrt{t}}{1-t} < 1\,,$$

which will hold when

$$\sqrt{|\omega(z)|} < \frac{2s}{1+\sqrt{1+4s^2}}\,, \quad s = \frac{1-t}{2\sqrt{2}C\sqrt{t}}\,.$$

□

We are finally in position to prove Theorem 1.1.

Proof. Without loss of generality we can assume that the unique critical point of $\mathcal{U}\widetilde{f}$ is the origin. Let $\Pi\colon \mathbb{R}^3 \to \mathbb{C}$ be the projection $\Pi(x_1, x_2, x_3) = x_1 + ix_2$. We know that $\overline{\Sigma} \cup \Sigma^*$ is locally a graph over $\mathbb{C}$, and hence the mapping

$$\mathcal{E}_f(\zeta) = \begin{cases} f(\zeta), & \zeta \in \overline{\mathbb{D}}, \\ (\Pi \circ \mathcal{R})(\widetilde{f}(\zeta^*)), & \zeta \notin \mathbb{D} \end{cases}$$

is locally injective.

Locating the critical point of $u_{\widetilde{f}}$ at the origin implies that $\mathcal{E}_f(z) \to \infty$ as $|z| \to \infty$. By the monodromy theorem we conclude that $\mathcal{E}_f$ is a homeomorphism of $\mathbb{C}$. In particular, the underlying harmonic mapping f is injective. Moreover, the assumption on ω implies that the inclinations of both Σ and Σ^* are bounded away from $\pi/2$, making the projection Π quasiconformal. Since the reflection $\mathcal{R}$ is quasiconformal, so is $\mathcal{E}_f$.

Let us verify that $\mathcal{E}_f$ has the stated form. For $z \in \mathbb{D}$ we have that $\Pi(\widetilde{f}) = f$, as claimed. For points outside $\mathbb{D}$ we recall (2.6) for the reflection

$$\mathcal{R} = \widetilde{f} + 2J(\nabla \log \lambda_\Sigma)\,,$$

so that

$$\Pi(\mathcal{R}) = f + \frac{2}{|\nabla \log \lambda_\Sigma|^2} J(\nabla \log \lambda_\Sigma)\,.$$

With the notation $X = e^{-\sigma}\widetilde{f}_x, Y = e^{-\sigma}\widetilde{f}_y$ used in (2.4) we have that

$$\nabla \log \lambda_\Sigma = X(\log \lambda_\Sigma)X + Y(\log \lambda_\Sigma)Y = e^{-\sigma}(\rho - \sigma)_x X + e^{-\sigma}(\rho - \sigma)_y Y\,,$$

so that

$$|\nabla \log \lambda_\Sigma| = e^{-2\sigma}\left([(\rho - \sigma)_x]^2 + [(\rho - \sigma)_y]^2\right) = 4e^{-2\sigma}\left|(\rho - \sigma)_z\right|^2\,,$$

and

$$\Pi(\nabla \log \lambda_\Sigma) = e^{-2\sigma}\left((\rho - \sigma)_x f_x + (\rho - \sigma)_y f_y\right) = 2e^{-2\sigma}\left((\rho - \sigma)_z h' + (\rho - \sigma)_z \overline{g'}\right)\,.$$

Putting these formulas together gives the desired result. □

5. Corollaries

The purpose of this section is to draw some corollaries from our main result by considering particular instances of the background metric $e^\rho|dz|$. For some of the applications we will use the following extension of Lemma 4 in [14].

Lemma 5.1. *Let $f = h + \bar{g}$ be a harmonic mapping defined in $\mathbb{D}$ with dilatation $\omega = q^2$ for some meromorphic q. If for some constant A*

$$|\mathcal{S}f| \leq \frac{2A}{(1 - |z|^2)^2} \tag{5.1}$$

then

$$\left|\sigma_z - \frac{\bar{z}}{1-|z|^2}\right| \le \frac{\sqrt{1+A}}{1-|z|^2}\,. \tag{5.2}$$

Proof. We briefly outline the proof and refer the reader to [14] for further details. If we let $\tau = |\sigma_z|$ then one can show from (5.1) that

$$|\tau_z| \ge \tau^2 - \frac{A}{(1-|z|^2)^2}\,.$$

We first prove the desired inequality at $z = 0$, that is, that $\tau(0) \le \sqrt{1+A}$. If not, then $|\tau_z|(0) > 0$ and we can consider an integral curve γ to $\nabla\tau$ starting at the origin. If $v(t) = \tau(\gamma(t))$, t an arclength parameter, then

$$v' \ge v^2 - \frac{A}{(1-|\gamma(t)|^2)^2} \ge v^2 - \frac{A}{(1-t^2)^2}\,.$$

Since the function $y = y(t)$ satisfying

$$y' = y^2 - \frac{A}{(1-t^2)^2}\ , \ \ y(0) = a$$

become infinite before time $t = 1$ when $a > \sqrt{1+A}$, a comparison gives the same for v, a contradiction. This proves the estimate at the origin, and the general case follows from linear invariance. □

For all our corollaries below, the choices of a complete conformal metric $e^\rho|dz|$ will satisfy conditions (ULP) and (BPJ) in light of Theorem 7 in [8]. The particular instance of that theorem we will require here to conclude both conditions is that $\rho_r \to \infty$ as $|z| \to 1$ and $|\rho_\theta|$ remains bounded.

Corollary 5.2. *Let $f = h + \bar{g}$ be a harmonic mapping defined in $\mathbb{D}$ with dilatation $\omega = q^2$ for some meromorphic q, and suppose that*

$$\left|\mathcal{S}f - \frac{2c(c-1)\bar{z}^2}{(1-|z|^2)^2}\right| + e^{2\sigma}|K| \le \frac{2tc}{(1-|z|^2)^2} \tag{5.3}$$

for some $0 \le t < 1$, $c > 1$, and that

$$\sup_{z\in\mathbb{D}} \sqrt{|\omega(z)|} < \frac{2s}{1+\sqrt{1+4s^2}}\ \ ,\ \ s = \frac{1-t}{2\sqrt{2}C\sqrt{t}}\,, \tag{5.4}$$

where $C = (c + \sqrt{1+c^2})/\sqrt{c}$. Then f is injective and has a quasiconformal extension to $\mathbb{C}$. given by

Proof. We apply the main theorem to the metric given by $\rho = -c\log(1-|z|^2)$. Because $c > 1$ the metric is complete. Straightforward computations show that (1.3) reduces to (5.3). We observe that condition (1.4) is met automatically for C

as stated in the corollary. This is so because, by the triangle inequality, the bound on the Schwarzian implies that

$$|Sf| \leq \frac{2c(c-1)+2tc}{(1-|z|^2)^2} \leq \frac{2c^2}{(1-|z|^2)^2}\,,$$

and the bound required in (1.4) follows now from Lemma 5.1 and the triangle inequality. This corollary can be considered a generalization of criteria obtained by Ahlfors in [2]. □

Corollary 5.3. *Let $f = h + \bar{g}$ be a harmonic mapping defined in $\mathbb{D}$ with dilatation $\omega = q^2$ for some meromorphic q, and suppose that*

$$4\left|\frac{z\sigma_z}{1-|z|^2}\right| + e^{2\sigma}|K| \leq \frac{2t}{(1-|z|^2)^2} \tag{5.5}$$

for some $0 \leq t < 1$, and that

$$\sup_{z\in\mathbb{D}} \sqrt{|\omega(z)|} < \frac{2s}{1+\sqrt{1+4s^2}} \quad , \quad s = \frac{1-t}{2\sqrt{2}\sqrt{t}}\,. \tag{5.6}$$

Then f is injective and has a quasiconformal extension to $\mathbb{C}$.

Proof. We will show that the family of dilations $f_r(z) = f(rz), r < 1$ are injective for in $\mathbb{D}$ and admit quasiconformal extensions with uniformly bounded distortion. A subsequence of these extension will converge to the desired quasiconformal mapping, giving in passing the injectivity of f. For fixed $r < 1$ we consider the metric $e^\rho|dz|$ given by $\rho = \sigma_r - \log(1-|z|^2)$, where e^{σ_r} is the conformal factor associated with the lift of f_r. Then $e^\rho|dz|$ is a complete metric of negative curvature, and (1.3) in Theorem 1.1 becomes

$$4(1-|z|^2)\,|rz\sigma_z(rz)| + 4r^2(1-|z|^2)^2\sigma_{z\bar{z}}(rz) \leq 2t\,,$$

which follows from (5.6) (evaluated ar rz) because $1-|z|^2 \leq 1-|rz|^2$. We also observe that condition (1.4) is met for the mapping f_r and the specified value of C because $|\sigma_z| \leq (t/2)(1-|z|^2)^{-1} \leq (1/2)(1-|z|^2)^{-1}$. The bounds on $||\omega_r||_\infty$ will also stay away from the critical value $2s/(1+\sqrt{1+4s^2})$, and we conclude that each f_r is injective in $\mathbb{D}$ and has a quasiconformal extension with uniformly bounded distortion. A convergent subsequence of these mappings will converge to the desired extension. □

One of the motivation of the present paper was to be able to obtain planar extensions of harmonic mappings satisfying criteria more general than what would correspond to the classical Nehari condition

$$|\mathcal{S}f| \leq \frac{2}{(1-|z|^2)^2}$$

for holomorphic f. Soon after discovering this criterion, Nehari found a family of new criteria involving a positive, continuous, even function $p = p(x)$ defined for $x \in (-1,1)$ for which

(i) $(1-x^2)^2 p(x)$ is non-increasing on $[0,1)$;

(ii) the equation $u'' + pu = 0$ is disconjugate, that is, non trivial solutions can vanish at most once in (-1,1).

Such a function will be called a Nehari function, for which Nehari proved

$$|\mathcal{S}f| \leq 2p(|z|)$$

was a sufficient condition for the univalence of f. The choice $p(x) = (1-x^2)^{-2}$ recovers the first case, while the choices $p(x) = 2(1-x^2)^{-1}$ and $\pi^2/4$ yield the interesting cases

$$|\mathcal{S}f| \leq \frac{4}{(1-|z|^2)} \quad \text{and} \quad |\mathcal{S}f| \leq \frac{\pi^2}{2}\,.$$

Let $\lambda = \lim_{x\to 1^-}(1-x^2)^2 p(x)$. Then $0 \leq \lambda \leq 1$ and $\lambda = 1$ if and only if $p(x) = (1-x^2)^{-2}$.

In [12] it was shown that

$$|\mathcal{S}f| + e^{2\sigma}|K| \leq 2p(|x|)$$

implied the injectivity of the lift $\widetilde{f}$ of a harmonic mapping f defined in $\mathbb{D}$ with dilatation a square. Our purpose will be to establish injectivity of f itself and a quasiconformal extension when

$$|\mathcal{S}f| + e^{2\sigma}|K| \leq 2\mu p(|z|)\,, \tag{5.7}$$

for $\mu < 1$. The case $p = (1-x^2)^{-2}$ was the subject of the paper [14], and therefore we will assume now that $\lambda < 1$. Choosing an appropriate complete conformal metric in Theorem 1.1 requires some preparation.

Let $u = u(x)$ be the solution of

$$u'' + pu = 0\,,\ u(0) = 1\,,\ u'(0) = 0\,.$$

Because u is even, it follows from the assumption of disconjugacy that u must remain positive on $(-1,1)$. Note also that u is decreasing on $[0,1)$, and gives rise to the (extremal) function

$$F(x) = \int_0^x u^{-2}(y)dy$$

that solves $SF = 2p, F(0) = 0, F'(0) = 1, F''(0) = 0$. The initial candidate for the metric is

$$u^{-2}(|z|)|dz|\,, \tag{5.8}$$

which will be always of negative curvature but will fail to be complete when $F(1) < \infty$. In Lemma 3 [9] it was shown that in this case there exists a maximal value $\tau_0 > 1$ such that $\tau_0 p$ remains a Nehari function for which the new extremal becomes infinite at $x = 1$. Because (5.7) trivially implies the same inequality for $\tau_0 p$ replacing p, there is no loss of generality in assuming that the metric in (5.8) is complete. One can also see that for $e^\rho = u^{-2}(|z|)$ one has

$$2\zeta^2(\rho_{zz} - \rho_z^2) = -A(|z|) + p(|z|) \quad , \quad 2\rho_{z\bar{z}} = A(|z|) + p(|z|)\,,$$

where $\zeta = z/|z|$ and

$$A(x) = \left(\frac{u'(x)}{u(x)}\right)^2 - \frac{1}{x}\frac{u'(x)}{u(x)}\,.$$

The inequality (1.3) in Theorem 1.1 would then read

$$\left|\zeta^2 \mathcal{S}f + A(|z|) - p(|z|)\right| + e^{2\sigma}|K| \le t(A(|z|) + p(|z|))\,. \tag{5.9}$$

Nevertheless, a second issue has to be resolved, namely that for given $\mu < 1$, (5.7) will not imply (5.9) for any $t < 1$ if $A/p \to \infty$ as $x \to 1$. It was shown in [9, Lemmas 2, 4] that $A(x) \ge p(x)$ in the complete case, and that the following limits exist:

$$L = \lim_{x \to 1^-} \frac{A(x)}{p(x)} = \frac{(1 + \sqrt{1-\lambda})^2}{\lambda}\,, \tag{5.10}$$

$$2\beta = -\lim (1 - x^2)\frac{u'(x)}{u(x)} = 1 + \sqrt{1-\lambda}\,. \tag{5.11}$$

Note that $L > 1$ because $\lambda < 1$ and that $L = \infty$ precisely when $\lambda = 0$. Observe also that $\beta > 1/2$. In order to overcome the difficulty that arises when $L = \infty$ one can perturb the compete metric (5.8) to

$$u^{-2\alpha}(|z|)|dz|\,, \tag{5.12}$$

for $\alpha < 1$ to be chosen appropriately. The resulting terms A_α, p_α are given by

$$Aa = \alpha^2\left(\frac{u'}{u}\right)^2 - \frac{\alpha}{x}\frac{u'}{u}\,,$$

$$p_\alpha = \alpha p + \alpha(1-\alpha)\left(\frac{u'}{u}\right)^2\,.$$

If the original limit $L = \infty$, one will now have

$$\lim_{x \to 1^-} \frac{A_\alpha(x)}{p_\alpha(x)} = \frac{\alpha}{1-\alpha} > 0\,,$$

which ensures that

$$l = \inf_{x \in [0,1)} \frac{A_\alpha(x)}{p_\alpha(x)} > 0\,.$$

Through Lemmas 5, 6 in [9] it was shown that there exists $\alpha_0 < 1$ close enough so that for all $\alpha \in [\alpha_0, 1)$ the perturbed metric is complete, $A_\alpha(x) \geq p_\alpha(x)$, and

$$t = \sup_{x\in[0,1)} \frac{2\mu p(x) + A_\alpha(x) - p_\alpha(x)}{A_\alpha(x) + p_\alpha(x)} = 1 - \frac{2\mu l}{1+l} < 1\,. \tag{5.13}$$

With this, we can finally state:

Corollary 5.4. *Let $f = h + \bar{g}$ be a harmonic mapping defined in $\mathbb{D}$ with dilatation $\omega = q^2$ for some meromorphic q, and let p be a Nehari function. Then for any $0 \leq \mu < 1$ there exists $c = c(p, \mu) > 0$ such that*

$$|\mathcal{S}f| + e^{2\sigma}|K| \leq 2\mu p(|z|) \tag{5.14}$$

and

$$\sup_{z\in\mathbb{D}} |\omega(z)| \leq c(p, \mu) \tag{5.15}$$

imply that f is injective in $\mathbb{D}$ and admits a quasiconformal extension.

Proof. The proof will show how the constant $c(p, \mu)$ is obtained. As we have seen, we may assume that the metric (5.8) is complete. If the limit L is finite we consider Theorem 1.1 and the choice of the unperturbed metric (5.8). Then (1.3) is given by

$$\left|\zeta^2\mathcal{S}f + A(|z|) - p(|z|)\right| + e^{2\sigma}|K| \leq t\left(A(|z|) + p(|z|)\right)\,,$$

which we claim is implied by (5.16) if $t = t(\mu)$ is chosen appropriately. Indeed, we have that

$$\begin{aligned}\left|\zeta^2\mathcal{S}f + A(|z|) - p(|z|)\right| + e^{2\sigma}|K| &\leq |\mathcal{S}f| + e^{2\sigma}|K| + A - p \\ &\leq (2\mu - 1)p + A \leq t(A + p)\end{aligned}$$

if we choose t so that

$$t = 1 - \frac{2\mu l}{1+l}\,.$$

We need to ensure that conditions (1.4) and (1.5) are met. As for (1.4), observe that

$$|\rho_z| = 2\left|\frac{u'}{u}\right| \leq 2\sqrt{A+p} = 2\sqrt{\rho_{z\bar{z}}}\,.$$

On the other hand since

$$|\mathcal{S}f| \leq |\mathcal{S}f| + e^{2\sigma}|K| \leq 2\mu p(|z|) \leq \frac{2\mu p(0)}{(1-x^2)^2}$$

we have from Lemma 5.1 that

$$|\sigma_z| \leq \frac{1 + \sqrt{1 + p(0)}}{1 - x^2}\,.$$

But

$$\inf_{x\in[0,1)} (1-x^2)^2(A(x) + p(x)) = \eta > 0$$

because $p(0) > 0$ and (5.11), therefore

$$|\rho_z| = 2\left|\frac{u'}{u}\right| \leq \frac{2}{\eta}\sqrt{\rho_{z\bar{z}}}\,.$$

The constant $\eta = \eta(p)$. We conclude that (1.4) is satisfied with

$$C = 1 + \sqrt{1+p(0)} + \frac{2}{\eta}\,.$$

The constant $c = c(p,\mu)$ is obtained now from condition (1.5).

If $L = \infty$ we need to consider (5.14) for α close but smaller than 1. With that, the rest of the argument is the same.

□

A particular case we would like to highlight is when $p = \pi^2/4$, so that (5.7) becomes

$$|\mathcal{S}f| + e^{2\sigma}|K| \leq \mu\frac{\pi^2}{2}\,.$$

The unperturbed metric is given by the conformal factor

$$e^{\rho} = u^{-2}(|z|) = \sec^2\left(\frac{\pi}{2}|z|\right)\,,$$

for which $L = \infty$. For the perturbed metric with conformal factor $e^{\alpha\rho}$, $\alpha < 1$, Theorem 1.1 takes the form

$$|\mathcal{S}f + A_\alpha - p_\alpha|\,| + e^{2\sigma}|K| \leq t(A_\alpha + p_\alpha)\,,$$

with a quasiconformal extension given by

$$\mathcal{E}_f(z) = f(z)$$

when $z \in \overline{\mathbb{D}}$ and

$$\mathcal{E}_f(z) = f(z^*) + \frac{2|z^*|h'}{\frac{\alpha\pi\bar{z}^*}{2}\tan\left(\frac{\pi}{2}|z^*|\right) - \frac{|z^*||h'|}{|h'|+|g'|}\frac{h''}{h'}} + \frac{2|z^*|\bar{g'}}{\frac{\alpha\pi z^*}{2}\tan\left(\frac{\pi}{2}|z^*|\right) - \frac{|z^*||g'|}{|h'|+|g'|}\overline{\frac{g''}{g'}}}$$

with $z^* = 1/\bar{z}$, when $z \notin \mathbb{D}$.

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